HOSSAM GHANEM

(29) 4.5 *Summary of Graphical Methods(B)*

Example 5 — 40 May 3, 2007

Let $f(x) = \dfrac{x^2-9}{2x-4}$ (8 pts.)

(a) Find the x and y-intercepts of f.

(b) Find the vertical and horizontal asymptotes to the graph of f, if any.

(c) Find the intervals on which f is increasing and the intervals on which f is decreasing , if any.

(d) Find the intervals on which f is concave up and the intervals on which f is concave down , if any.

(e) Sketch the graph of f.

Solution

(a) x –intercepts

at $y = 0 \Rightarrow \quad x^2 - 9 = 0 \quad \Rightarrow x = \pm 3$

$(-3,0) \quad , \quad (3,0)$

y –intercept

at $x = 0 \Rightarrow \quad f(0) = \dfrac{-9}{-4} = \dfrac{9}{4}$

$\left(0, \dfrac{9}{4}\right)$

(b)

$H.A$:

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x^2-9}{2x-4} = \infty \qquad (No \quad H.A)$$

$V.A$:

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} \frac{x^2-9}{2x-4} = \infty$$

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} \frac{x^2-9}{2x-4} = -\infty$$

$\therefore \quad x = 2 \qquad V.A$

(c)

$$f(x) = \frac{x^2-9}{2x-4}$$

$$f^\backslash(x) = \frac{(2x-4)(2x)-(x^2-9)(2)}{(2x-4)^2} = \frac{4x^2 - 8x - 2x^2 + 18}{4(x-2)^2} = \frac{2x^2 - 8x + 18}{4(x-2)^2} = \frac{2(x^2-4x+9)}{4(x-2)^2}$$

$B^2 - 4AC = 16 - 4(9) < 0$

$f^{\backslash}(x) > 0 \quad \Rightarrow \quad \therefore f \nearrow \; on\ \mathcal{R}/\{2\}$ $\qquad f^{\backslash}(x) \neq 0 \quad \Rightarrow$ No local extrema

(d)

$$f^{\backslash}(x) = \frac{x^2 - 4x + 9}{2(x-2)^2}$$

$$f^{\backslash\backslash}(x) = \frac{2(x-2)^2(2x-4) - (x^2 - 4x + 9)\cdot 4(x-2)}{4(x-2)^4} = \frac{2(x-2)^2 \cdot 2(x-2) - 4(x-2)(x^2 - 4x + 9)}{4(x-2)^4}$$

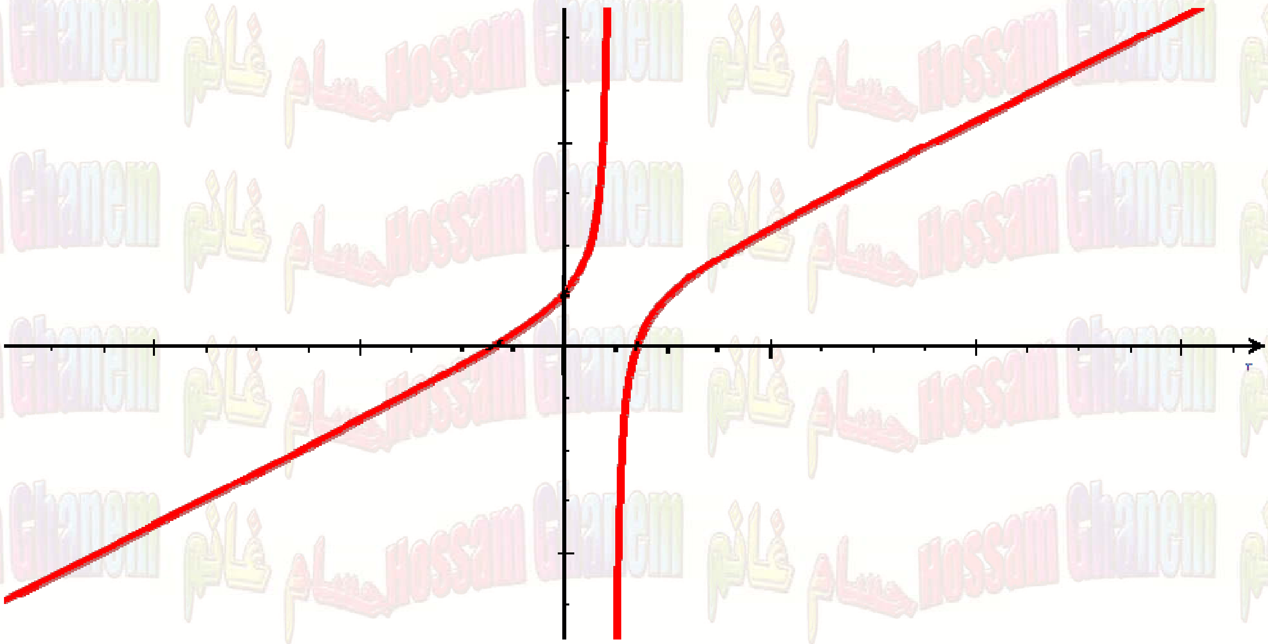
$$= \frac{4(x-2)^3 - 4(x-2)(x^2 - 4x + 9)}{4(x-2)^4} = \frac{4(x-2)[(x-2)^2 - (x^2 - 4x + 9)]}{4(x-2)^4}$$

$$= \frac{(x-2)[x^2 - 4x + 4 - x^2 + 4x - 9]}{(x-2)^4} = \frac{-5}{(x-2)^3}$$

The graaph f CU on $(-\infty, 2)$
The graaph f CD on $(2, \infty)$
$f^{\backslash\backslash}(x) \neq 0 \quad \Rightarrow$ No inflection points

$-(x-2)^3$	$+$	2	$-$
	CU		CD

(e)



Example 6

41 July 19, 2007

Let $f(x) = \dfrac{x^2+1}{x}$ [9 pts.]

(a) Find the vertical and horizontal asymptotes for the graph of f, if any.
(b) Find the intervals on which f is increasing or decreasing and find local extrema, if any.
(c) Find the intervals on which the graph of f is concave upward or concave downward and find the points of inflection, if any.
(d) Discuss the symmetry of the graph.
(e) Sketch the graph of f.

Solution

(a)

$H.A:$

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x^2+1}{x} = \infty$$

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{x^2+1}{x} = -\infty$$

$No \quad H.A$

$V.A:$

$$\lim_{x\to0^-} f(x) = \lim_{x\to0^-} \frac{x^2+1}{x} = -\infty$$

$$\lim_{x\to0^+} f(x) = \lim_{x\to0^+} \frac{x^2+1}{x} = \infty$$

$\therefore \; x = 0 \quad V.A$

(b)

$$f(x) = \frac{x^2+1}{x}$$

$$f^\backslash(x) = \frac{x(2x)-(x^2+1)}{x^2} = \frac{2x^2-x^2-1}{x^2} = \frac{x^2-1}{x^2} = \frac{(x-1)(x+1)}{x^2}$$

$f \;\nearrow$ on $(-\infty,-1)\cup(1,\infty)$

$f \;\searrow$ on $(-1,1)\,/\{0\}$

$f^\backslash(x) = 0$
$(x-1)(x+1) = 0$

$x = -1 \qquad f(-1) = \dfrac{1+1}{-1} = -2$

$x = 1 \qquad f(1) = \dfrac{1+1}{1} = 2$

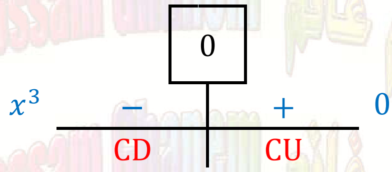
		−1		0		1		
$x-1$	−		−		−		+	1
$x+1$	−		+		+		+	−1
x^2	+		+		+		+	0
	⊕		⊖		⊖		⊕	
	↗		↘		↘		↗	

$\therefore$ Maximum local extrema at $(-1,-2)$

Minimum local extrema at $(1,2)$

$$f^{\backslash}(x) = \frac{x^2 - 1}{x^2}$$

$$f^{\backslash\backslash}(x) = \frac{x^2(2x) - (x^2 - 1)(2x)}{x^4} = \frac{2x(x^2 - x^2 + 1)}{x^4} = \frac{2x}{x^4} = \frac{2}{x^3}$$



The graph of f CD on $(-\infty, 0)$
The graph of f CU on $(0, \infty)$

$f^{\backslash\backslash}(x) \neq 0$
No inflection points

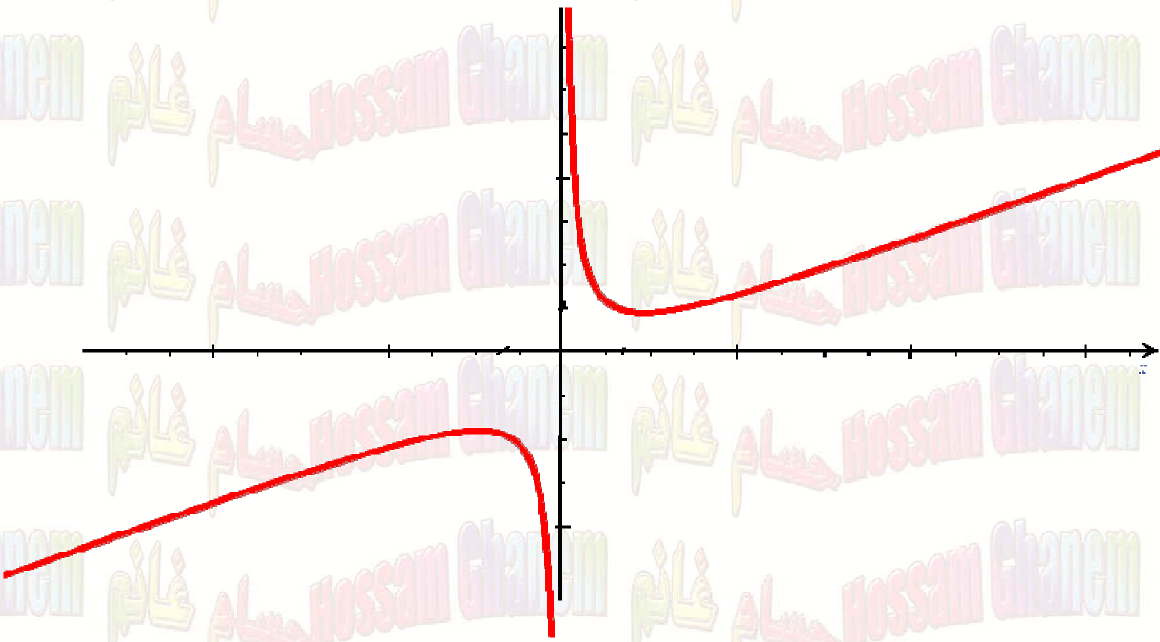
(d)

$$f(x) = \frac{x^2 + 1}{x}$$

$$f(-x) = \frac{x^2 + 1}{-x} = -\frac{x^2 + 1}{x} = -f(x)$$

$\therefore$ the graph semmetric to the orgin

(e)



Example 7

42 May 5, 2008

Let $f(x) = \dfrac{x}{1-x^2}$ and given that $f'(x) = \dfrac{1+x^2}{(1-x^2)^2}$ and $f''(x) = \dfrac{2x(x^2+3)}{(1-x^2)^3}$

(a) Find the vertical and horizontal asymptotes for the graph of f, if any.

(b) Find the intervals on which f is increasing and the intervals on which f is decreasing. Find the local extrema of f, if any.

(c) Find the intervals on which the graph of f is concave upward and the intervals on which the graph of f is concave downward. Find the points of inflection, if any.

(d) Sketch the graph of f.

Solution

(a)

$H.A$:

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x}{1-x^2} = 0$$

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{x}{1-x^2} = 0$$

$$\therefore \ y = 0 \qquad H.A$$

$V.A$:

$$f(x) = \frac{x}{1-x^2} = \frac{-x}{x^2-1} = \frac{-x}{(x-1)(x+1)}$$

$$\lim_{x\to-1^-} f(x) = \lim_{x\to-1^-} \frac{-x}{(x-1)(x+1)} = \infty$$

$$\lim_{x\to-1^+} f(x) = \lim_{x\to-1^+} \frac{-x}{(x-1)(x+1)} = -\infty$$

$$\therefore \ x = -1 \qquad V.A$$

$$\lim_{x\to1^-} f(x) = \lim_{x\to1^-} \frac{-x}{(x-1)(x+1)} = \infty$$

$$\lim_{x\to1^+} f(x) = \lim_{x\to1^+} \frac{-x}{(x-1)(x+1)} = -\infty$$

$$\therefore \ x = 1 \qquad V.A$$

(b)

$$f^{\backslash}(x) = \frac{x^2+1}{(1-x^2)^2}$$

$f^{\backslash}(x) > 0$

$f \nearrow$ on $R/\{-1,1\}$

$f^{\backslash}(x) \neq 0$

No local extrema

(c)

$$f^{\backslash\backslash}(x) = \frac{2x(x^2+3)}{(1-x^2)^3} = \frac{2x(x^2+3)}{(1-x)^3(1+x)^3}$$

		-1		0		1		
x	$-$		$-$		$+$		$+$	0
$(1-x)^3$	$+$		$+$		$+$		$-$	1
$(1+x)^3$	$-$		$+$		$+$		$+$	5
	$\oplus$ CU		$\ominus$ CD		$\oplus$ CU		$\ominus$ CD	

The graph of f CU on $(-\infty,-1) \cup (0,1)$

The graph of f CD on $(-1,0) \cup (1,\infty)$

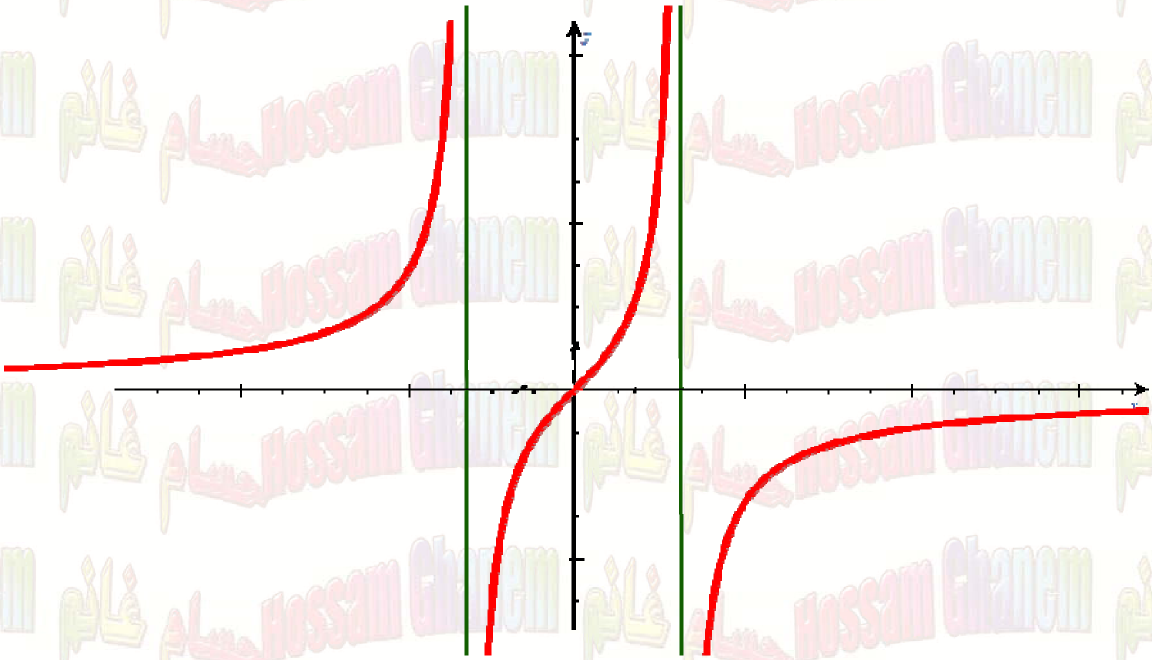
$f^{\backslash\backslash}(x) = 0$

$x = 0$

$$f(0) = \frac{0}{1-0} = 0$$

at (0 , 0) inflection point

(*d*)



Example 8

43 July 19, 2008

Let $f(x) = \dfrac{-2x}{x+1}$ .and given that $f'(x) = \dfrac{-2}{(x+1)^2}$ and $f''(x) = \dfrac{4}{(x+1)^3}$

(a) Find the vertical and horizontal asymptotes for the graph of f, if any.

(b) Find the intervals on which f is increasing and the intervals on which f is decreasing. Find the local extrema of f, if any.

(c) Find the intervals on which the graph of f is concave upward and the intervals on which the graph of f is concave downward. Find the points of inflection, if any.

(d) Sketch the graph of f.

(e) Find the maximum and the minimum values of on $[1, 2]$.

(10 points)

Solution

(a)

$H.A$:

$$\lim_{x\to\infty} \frac{-2x}{x+1} = -2$$

$$\lim_{x\to-\infty} \frac{-2x}{x+1} = -2$$

$\therefore\ y = -2 \quad H.A$

$V.A$:

$$\lim_{x\to-1^-} f(x) = \lim_{x\to-1^-} \frac{-2x}{x+1} = -\infty$$

$$\lim_{x\to-1^+} f(x) = \lim_{x\to-1^+} \frac{-2x}{x+1} = \infty$$

$\therefore\ x = -1 \quad V.A$

$$f'(x) = \frac{-2}{(x+1)^2}$$

$f'(x) < 0$ on $R/\{-1\}$

f ↘ on $R/\{-1\}$

$f'(x) \neq 0$

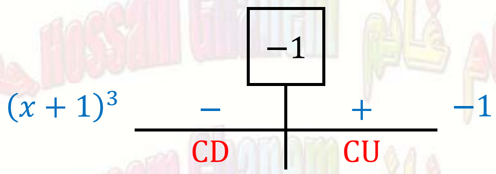
No local extrema

(c)

$$f''(x) = \frac{4}{(x+1)^3}$$

The graph of f CD on $(-\infty, -1)$

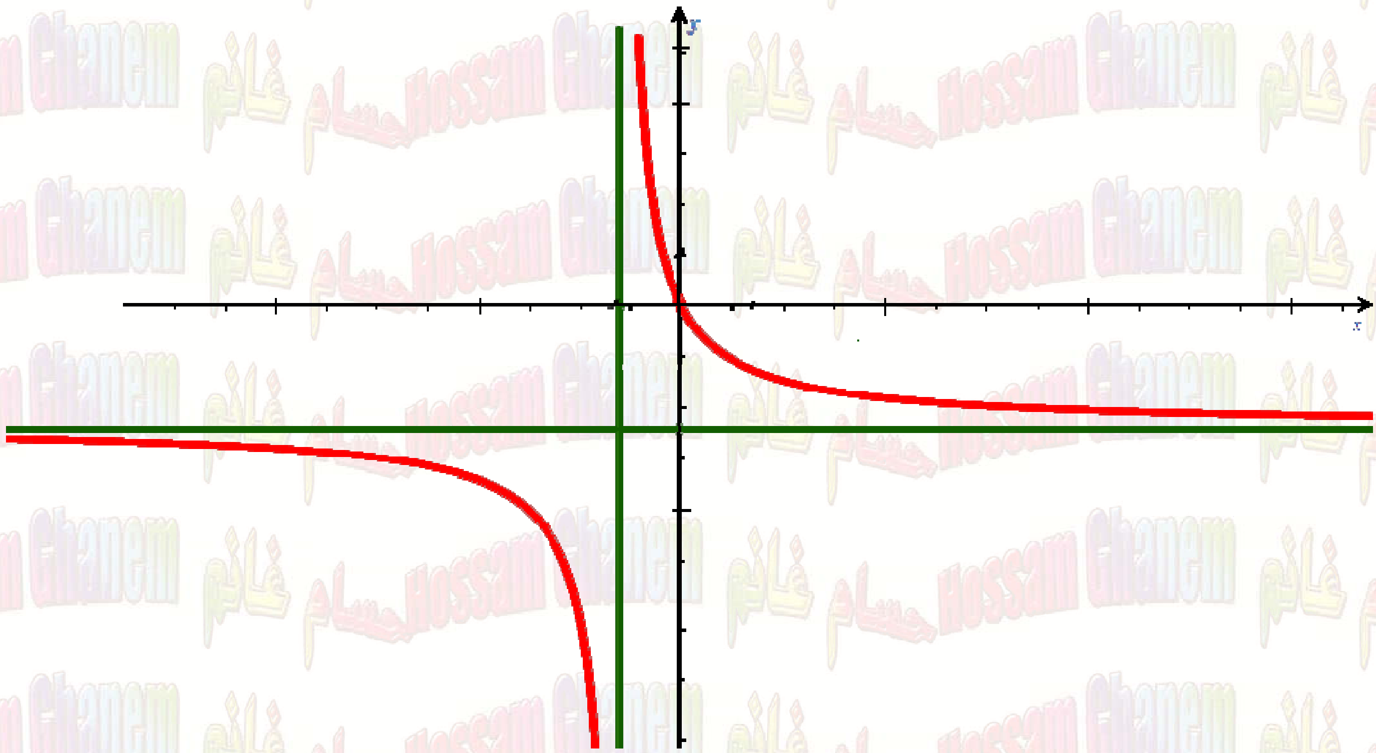
The graph of f CU on $(-1, \infty)$



$f''(x) \neq 0$

No inflection point

(d)



(e)

$f(1) = \frac{-2}{2} = -1$ *Max*

$f(2) = \frac{-4}{3}$ *Min*

Homework

1

25 December 10, 2000

Let $f(x) = 3(2+x)\sqrt[3]{x}$ and note that $f^{\backslash}(x) = \dfrac{4x+2}{\sqrt[3]{x^2}}$ and $f^{\backslash\backslash}(x) = \dfrac{4x-4}{3x\sqrt[3]{x^2}}$

Answer the following (1 point each)

(a) Show that the graph of f has a vertical tangent
(b) Determine the intervals on which f is increasing, and the intervals on which f is decreasing.
(c) What are the local extrema of f, if any?
(d) Determine the intervals on which the graph of f is concave upward and the intervals on which the graph is concave downward.
(e) What are the points of inflection of the graph of f, if any?
(f) Sketch the graph of f indicating vertical tangents, local extrema, concavity, points of inflection, and asymptotes, if any

2

29 July 25th, 2002

Let $f(x) = \dfrac{x^2}{x^2-1}$, $f^{\backslash}(x) = \dfrac{-2x}{(x^2-1)^2}$ and $f^{\backslash\backslash}(x) = \dfrac{6x^2+2}{(x^2-1)^3}$

a) Find the vertical and horizontal asymptotes (if any).

b) Find the intervals on which f is increasing and the intervals on which f is decreasing. Find the local extrema (if any).

c) Find the intervals on which the graph of f concave upward and the intervals on which the graph of f is concave downward. Find the point of inflection (if any).

d) Discuss the symmetries of the graph of f.

e) Sketch the graph of f.